Fermions at unitarity and Haldane exclusion statistics

R.K. Bhaduri[1,2], M.V.N. Murthy[2] and M.K. Srivastava[3]

[1] *Department of Physics and Astronomy, McMaster University, Hamilton, Canada L8S 4M1*

[2] *The Institute of Mathematical Sciences, Chennai 600113, India*

[3] *Institute Instrumentation Centre, Indian Institute of Technology, Roorkee 247667, India*

PACS. 03.75.Ss – Degenerate Fermi gases.
PACS. 05.30.-d – Quantum statistical mechanics.

Abstract. –

We consider a gas of neutral fermionic atoms at ultra-low temperatures, with the attractive interaction tuned to Feshbach resonance. We calculate, the variation of the chemical potential and the energy per particle as a function of temperature by assuming the system to be an ideal gas obeying the Haldane-Wu fractional exclusion statistics. Our results for the untrapped gas compare favourably with the recently published Monte Carlo calculations of two groups. For a harmonically trapped gas, the results agree with experiment, and also with other published work.

Consider a dilute gas of neutral fermionic atoms at a low temperature. In general, the low-energy properties of the gas are determined by the scattering length a, the number density n, and the temperature T of the gas (the effective range r_0 is small, so that $r_0/|a| \to 0$ as a becomes large). When the attractive interaction between the atoms is increased continuously by magnetic tuning from weak to strong, the scattering length a goes from a small negative to a small positive value. In between, there is a zero-energy two-body bound state, and $|a|$ is infinite. The gas is said to be at unitarity in this situation, and the length scale a drops out. The behaviour of the gas is expected to be universal at unitarity [1]. Experimentally, if the temperature is small enough, a BCS superfluid is observed at the weak end, and a BEC condensate of dimers at the strong end [2]. This was predicted long back by Leggett [3], who extended the BCS formalism in a novel fashion to analyse the physical situation. The BCS to BEC transition is found to be smooth, with no discontinuity in properties across the unitary point. There has been much interest amongst theorists to calculate the properties of the gas in the unitary regime ($k_f|a| >> 1$), where $k_f = (3\pi^2 n)^{1/3}$ is the Fermi wave number of the noninteracting gas. This is a challenging task, since there is no small expansion parameter, and a perturbative calculation cannot be done. In particular, at $T = 0$, the energy per particle of the gas is calculated to be

$$\frac{E}{N} = \xi \, \frac{3}{5} \frac{\hbar^2 k_f^2}{2M}, \tag{1}$$

where $\xi \simeq 0.44$ [4]. The experimental value is about 0.5, but with large error bars [5]. Recently, there have been two Monte Carlo (MC) finite temperature calculations [6, 7] of an untrapped gas at unitarity, where various thermodynamic properties as a function of temperature have been computed. For a harmonically trapped gas, there are experimental results [8], as well as theoretical calculations [9].

In the unitary regime, the thermodynamic properties have both bosonic and fermionic features [6], and it is natural to ask if in this regime the quasi-particles obey a statistics which is intermediate between the two. In this paper, we suggest on general grounds that at unitarity, so far as average properties of the system are concerned, it should behave like an ideal gas obeying the generalised exclusion statistics of Haldane [10]. The definition of the statistical parameter, denoted by $g(> 0)$ in the present paper, is based on the rate at which the number of available states in a system of fixed size deceases as more and more particles are added to it. The statistical parameter g assumes values 0 and 1 for bosons and fermions respectively, because the addition of one particle reduces the number of available states by g. We first deduce the value of g for the unitary gas from theory using Eq.(1), fitting $\xi = 0.44$. The value of g thus determined remains the same independent of the nature of confinement as it should since the microscopic origin of the value of g depends only on the interaction between fermions and not on how the system is prepared. The application of the finite temperature distribution function [11] (called Haldane-Wu statistics in this paper) then enables us to calculate the temperature dependence of the energy per particle, and the chemical potential of the unitary gas. Our results for both trapped and untrapped gases are in good agreement with experiment, and MC calculations.

We shall now give the rationale for using Haldane-Wu statistics at unitarity:

- A strong hint that this may be the case comes from the observation that the kinetic and potential energies scale the same way when there is no length scale left from the interaction. As is well known, Haldane-Wu statistics is realised by the Calogero-Sutherland model in one dimension [13]. The potential and kinetic energy both scale the same way in this model, and both the energy densities scale as n^3. Similarly, fermions in two dimensions interacting with a zero-range potential have their kinetic and potential energy densities scale as n^2, obeying Haldane-Wu statistics [14].

- In the present case, a compelling evidence comes from the fact that the second virial coefficient of the gas at unitarity is temperature independent [15]. In exclusion statistics, the scale-invariant interaction between atoms alters the ideal Fermi (Bose) values of the (exchange) second virial coefficient $+(-)2^{-5/2}$ by adding an interacting part [16].

The above arguments are heuristic and indicative. A quantitative understanding can be obtained only when the effective interaction is known fully. In the absence of such a theory, in this paper we pursue a phenomenological approach where we assume the validity of exclusion statistics on the average for quasi-particles which are otherwise non-interacting. The effect of interaction is entirely subsumed in defining the statistics of the quasi-particles. We first estimate the value of the statistical parameter g from the following considerations:

For Haldane-Wu statistics, the distribution function (or occupancy factor) in a single particle state with energy ϵ_i is given by $f_i = (w_i + g)^{-1}$, where w_i obeys the relation

$$w_i^g (1 + w_i)^{1-g} = exp[(\epsilon_i - \mu)\beta] \ , \tag{2}$$

where $\beta = 1/T$, T being the temperature in units of the Boltzmann constant. Note from the above that for $g = 0$ and 1, the distribution function f_i reduces to the familiar bosonic and

fermionic forms. It is also clear that for $T = 0$, the occupancy factor is

$$\begin{aligned} f_i(T=0) &= \frac{1}{g}\ , \quad \epsilon_i < \mu, \\ f_i(T=0) &= 0\ , \quad \epsilon_i > \mu\ . \end{aligned} \tag{3}$$

Now consider N fermionic atoms obeying this statistics at $T = 0$ in a large volume V. The new Fermi momentum $\tilde{k_f}$ is determined from the relation

$$N = V\frac{1}{g}\frac{2}{(2\pi)^3}\int_0^{\tilde{k_f}} 4\pi k^2 dk\ ,$$

where we have included a spin degeneracy factor of 2. The modified Fermi momentum $\tilde{k_f}$, from above. is $\tilde{k_f} = g^{1/3}k_f$, where k_f is the fermi momentum of the noninteracting Fermi gas. It also follows that the energy per particle of the unitary gas is given by

$$\frac{E}{N} = g^{2/3}\ \frac{3}{5}\frac{\hbar^2 k_f^2}{2M}\ .$$

Comparing with Eq.(1), we see that $\xi = g^{2/3}$, and choosing $g = 0.29$ gives the generally accepted value of $\xi = 0.44$. This therefore fixes the only free parameter in the model, namely, g and it should be valid independent of temperature and the nature of confinement as it is the parameter which determines the statistics of quasi-particles.

The main advantage of our model, however, is the calculation of the bulk properties of the gas as a function of the temperature, and this we proceed to do now. We follow the well known method (see for example the paper by Aoyama [17]) for this purpose. For a given density of single-particle states $D(\epsilon)$, we have

$$N = \int_0^\infty \frac{D(\epsilon)d\epsilon}{(w+g)}, \quad E = \int_0^\infty \frac{\epsilon D(\epsilon)d\epsilon}{(w+g)}. \tag{4}$$

For the 3−dimensional gas, $D(\epsilon) = C\sqrt{\epsilon}$, where the constant $C = \frac{3}{2}N\epsilon_f^{-3/2}$. Furthermore, $\epsilon_f = \frac{\hbar^2 k_f^2}{2M}$ is the Fermi energy of the noninteracting Fermi gas. Changing the variable from $d\epsilon$ to dw, and using the relation involving w's given above, one gets after some algebra

$$\frac{3}{2}\left(\frac{T}{\epsilon_f}\right)^{3/2}\int_{w_0}^\infty \frac{dw}{w(1+w)}\left[\ln\left\{\left(\frac{w}{w_0}\right)^g\left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^{1/2} = 1, \tag{5}$$

$$\frac{E}{N\epsilon_f} = \frac{3}{2}\left(\frac{T}{\epsilon_f}\right)^{5/2}\int_{w_0}^\infty \frac{dw}{w(1+w)}\left[\ln\left\{\left(\frac{w}{w_0}\right)^g\left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^{3/2}. \tag{6}$$

In the above, w_0 is the value of w at $\epsilon = 0$. For our choice of $g = 0.29$, the Eq. (5) is solved at a given (T/ϵ_f) for w_0 numerically, and this w_0 is used in Eq.(6) next to obtain $(E/N\epsilon_f)$. From the definition of w_0, it also follows that the chemical potential μ at temperature T obeys the relation

$$\frac{\mu}{\epsilon_f} = -\frac{T}{\epsilon_f}\left[g\ln w_0 + (1-g)\ln(1+w_0)\ \right]. \tag{7}$$

Our results for the energy per particle and the chemical potential (in units of the noninteracting Fermi energy ϵ_f) are plotted in Fig.1 and Fig.2 respectively.

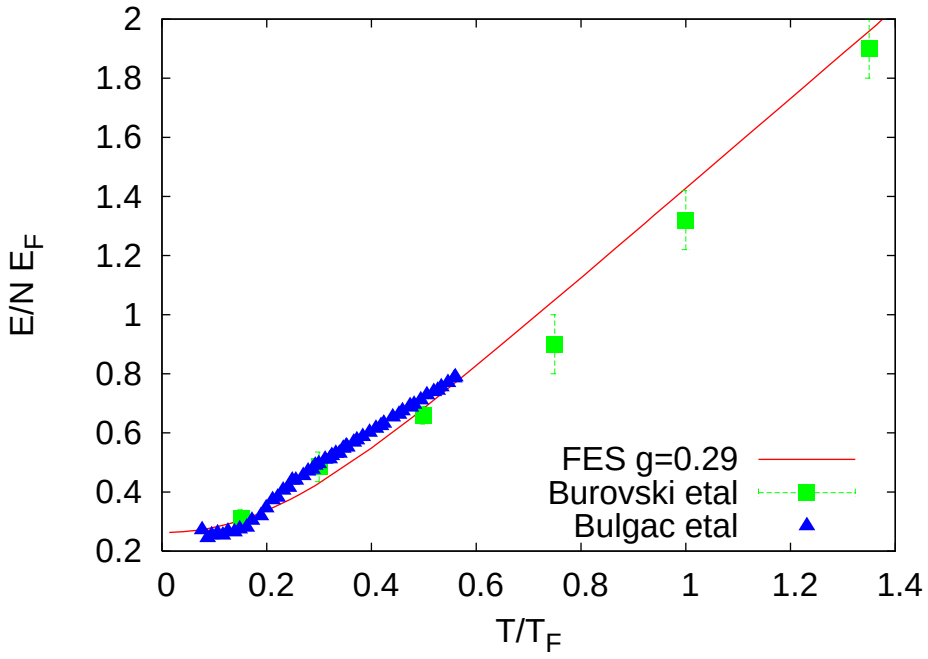

Fig. 1 – Plot of the energy per particle as a function of temperature. Both the abscissa and ordinate are in units of the free Fermi energy. The solid line corresponds to our calculations with $g = 0.29$. The solid squares (green) with error bars are the MC calculations of [7], and the triangles (blue) are the MC calculations of [6].

Our results are not sensitive to the fine-tuning of the statistical parameter g. In Fig. 1, we also show, for comparison, the recent MC calculated points of Bulgac *et al.* [6] and Burovski *et al.* [7]. It will be seen that the agreement is very good, although the chemical potential μ as calculated by us starts to differ from Burovski *et al* result for $T/\epsilon_f > 0.8$).

The finite temperature results are easily generalised for fermions in harmonic trap. Consider the fermions at $T = 0$. The density of states $D(\epsilon)$, including a spin degeneracy factor of 2, is $\epsilon^2/(\hbar\omega)^3$, where the oscillator parameter is defined as $\omega = (\omega_x\omega_y\omega_z)^{1/3}$. It follows immediately that $\tilde{\epsilon_f} = (3gN)^{1/3}\hbar\omega$, and the energy $E = g^{1/3}(3N)^{4/3}/4\ \hbar\omega$. These results are the same as the Thomas-Fermi density functional approach of Papenbrock [18]. We can easily extend these results to finite temperatures using this density of states in Eq.(4).

$$1 = 3\left(\frac{T}{\epsilon_f}\right)^3 \times \int_{w_0}^{\infty} \frac{dw}{w(1+w)} \left[\ln\left\{\left(\frac{w}{w_0}\right)^g \left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^2, \tag{8}$$

$$\frac{E}{N\epsilon_f} = 3\left(\frac{T}{\epsilon_f}\right)^4 \times \int_{w_0}^{\infty} \frac{dw}{w(1+w)} \left[\ln\left\{\left(\frac{w}{w_0}\right)^g \left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^3. \tag{9}$$

The expression for μ remains the same as Eq.(7), although the numerical values of w_0 as a function of T are quite different from the unconfined gas. We present the results for average energy in the trap in Fig.3, using the same value of $g = 0.29$ since the statistical parameter depends only on the mutual interaction and not on the nature of confinement. It will be

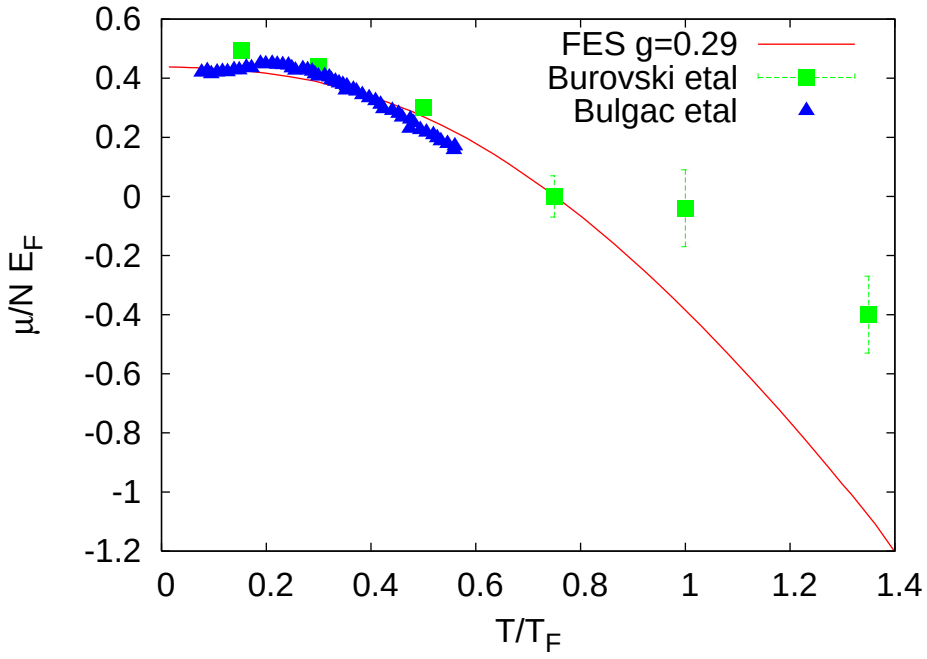

Fig. 2 – Same as in Fig.(1) for chemical potential plotted as a function of temperature.

seen that the agreement with experimental data of Kinast et al [8] as well as the many body calculation of Hu et al [9] is very good.

Other thermodynamic quantities could be readily calculated. However we note that the model cannot yield the two- or many-particle correlation functions. In this regard, the situation is similar to the one-dimensional Calogero-Sutherland model [12], which can be mapped on to a system of quasi-particles which obey Haldane-Wu statistics [13]. But this does not help in obtaining the correlation functions, for which the full many-body calculation has to be done. Moreover, the ideal Haldane-Wu gas cannot describe super-fluidity. Therefore, the main usefulness of the present approach is its ability to calculate the temperature-dependence of various bulk properties of a unitary gas with just one free parameter, namely the statistical parameter g.

* * *

We are extremely grateful to Evgeni Burovski, Joaquin Drut and Hui Hu for sharing the results of their calculations with us. We would like to thank Akira Suzuki for useful discussions, and for invaluable help in preparing the manuscript. One of the authors (R.K.B.) benefited greatly from many discussions with Duncan O'Dell. Thanks are also due to Jules Carbotte. This research was supported by NSERC of Canada.

REFERENCES

[1] Baker G.A., Phys. Rev. **C60**, 054311 (1999); Heiselberg H., Phys. Rev. **A63**, 043606 (2001); Ho T.-L., Phys. Rev. Lett. **92**, 090402 (2004).

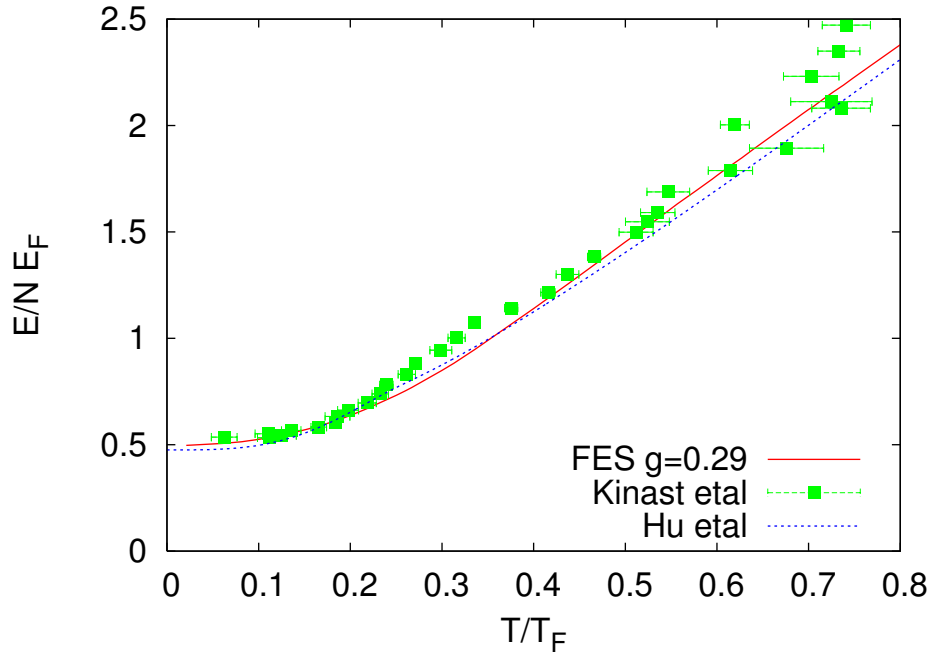

Fig. 3 – Plot of the energy per particle as a function of temperature for the confined gas. Both the abscissa and ordinate are in units of the free Fermi energy. The solid line corresponds to our calculations with $g = 0.29$ and the dashed lines corresponds to the calculations presented in Ref. [9]. The experimental data is taken from Kinast et al [8].

[2] Regal C.A. *et al.*, Nature (London) **424**, 47 (2003); Zwierlein M.W. *et al.*, Phys. Rev. Lett. **91**, 250401 (2003); Regal C.A. *et al.*, Phys. Rev. Lett. **92**, 040403 (2004); Zwierlein M.W. *et al.*, Nature (London) **435**, 1046 (12005); Partridge G.B.*et al.*, Science **311**, 503 (2006).

[3] Leggett A.J., in *Modern Trends in the Theory of Condensed Matter*, Springer-Verlag Lecture Notes, Vol. 115, edited by Peklaski A and Przystawa J.,(Springer-Verlag, Berlin, 1980), p.13

[4] Carlson J., Chang S.-Y., Pandharipande V.R., and Schmidt K.E., Phys. Rev. Lett. **91**, 050401 (2003); Perali A., Pieri P., and Strinati G.C., Phys. Rev. Lett. **93**, 100404 (2004).

[5] Bartenstein M. *et al.*, Phys. Rev. Lett. **92**, 120401 (2004); Bourdel T. *et al.*, Phys. Rev. Lett. **93**, 050401 (2004).

[6] Bulgac A., Drut J.E., and Magierski P., Phys. Rev. Lett. **96**, 090404 (2006).

[7] Burovski E., Prokof'ev N., Svistunov B., and Troyer M., Phys. Rev. Lett. **96**, 160402 (2006).

[8] Kinast J., Turlapov A., Thomas J.E., Qijin Chen, Jelena Stajic and Levin K., Science **307** 1296 (2005).

[9] Hu H.. Xia-Ji Lu and Drummond D., Phys. Rev. **A73**, 023617 (2006).

[10] Haldane F.D.M., Phys. Rev. Lett. **67**, 937 (1991).

[11] Dasnieres de Veigy A., and Ouvry S., Phys. Rev. Lett. **72**, 600 (1994); Wu Y.-S., Phys. Rev. Lett. **73**, 922 (1994); Isakov S.B., Phys. Rev. Lett. **73**, 2150 (1994); Rajagopal A.K.,Phys. Rev. Lett. **74**, 1048 (1995).

[12] Calogero F., J. Math. Phys. **10**, 2191 (1969); **10**, 2197 (1969); Sutherland B., J. Math. Phys. **12**. 246 (1971); **12**, 251; Phys. Rev. **A4**, 2019 (1971).

[13] Ha Z.N.C., Phys. Rev. Lett.**73**, 1574 (1994); Isakov S.B., Phys. Rev. Lett. **73**, 2150 (1994); Murthy M.V.N., and Shankar R., Phys. Rev. Lett. **73**, 3331 (1994).

[14] Bhaduri R.K., Murthy M.V.N., and Srivastava M.K., Phys. Rev. Lett. **76**, 165 (1996); Srivastava

M.K., Bhaduri R.K., Law J., and Murthy M.V.N., Can. J. Phys., **78**9 (2000).
[15] Ho T.-L., and Mueller E.J., Phys. Rev. Lett. **92**, 160404 (2004).
[16] Murthy M.V.N., and Shankar R., Phys. Rev. Lett. **72**, 3629 (1994).
[17] Aoyama T., Eur. Phys. J. **B 20**, 123 (2001); cond-mat/0005336 v2
[18] Papenbrock T., Phys. Rev. **A72**, 041603 (2005).